A General Option Valuation Approach to Discount for Lack of Marketability

Robert Brooks, PhD

A general option-based approach to estimating the discount for lack of marketability is offered. It is general enough to capture maturity, volatility, hedging availability, and investor skill, as well as other important factors. The model is shown to contain several option-based models as special cases. The model also contains two weighting variables that provide valuation professionals much needed flexibility in addressing the unique challenges of each nonmarketable valuation assignment. Selected prior empirical results are reinterpreted with this approach.

Introduction

Many financial instruments are not marketable, such as private equity, restricted stock in publicly traded companies, and many over-the-counter financial derivatives contracts. This lack of marketability normally results in a lower value when compared to an otherwise equivalent publicly traded instrument. Assigning an appropriate discount for lack of marketability (DLOM) is a challenging task requiring both quantitative analysis and qualitative professional judgment. By definition, any empirical analysis of DLOM will involve the difficult problem of positing an appropriate valuation model for the nonmarketable instrument. The purpose of this paper is not to resolve this difficult valuation model problem; the purpose is to offer a new approach for the financial professional seeking to address this challenging task.

Unlike prior models, the model presented here is general enough to capture maturity, volatility, hedging availability, and investor skill, as well as other important factors, and yet it is easy to interpret and use. Four major contributions are provided here: First, Longstaff’s (1995) model is decomposed into two separable components that can characterize whether the nonmarketable instrument can be hedged and whether the nonmarketable instrument owner possesses any sort of skill related to this particular instrument (for example, market timing ability). Second, Finnerty’s (2012) model is reinterpreted in a manner that can be easily justified in practice. Third, Barenbaum, Schubert, and Garcia’s (2015) model is also reinterpreted within this new model. Finally, a general option valuation–based model is introduced that provides an intellectually rigorous framework yet remains flexible enough to be applied in practice.

According to Abrams (2010), the first tenuous evidence of DLOM can be found in the sale of Joseph for twenty pieces of silver when the going rate was thirty, i.e., a discount of 33%. Stockdale (2013, 13) identifies an unnamed federal income tax case in 1934 as the first mention of DLOM. From 1934 through the 1970s, the admissible DLOM averaged between 20% and 30%.

Based on a 1969 Securities and Exchange Commission (SEC 1969) document, mutual funds held in excess of $3.2 billion of restricted equity securities, or about 4.4% of their total net assets. The SEC recognized that mutual fund managers would be tempted to report the net asset value of these restricted securities at or near the current publicly traded price, creating an instant gain upon which the manager would be compensated. For example, suppose a company with $100 publicly traded stock offered a mutual fund restricted shares at $75. Upon acquisition, the mutual fund manager would be tempted to assert the value of the restricted shares at $100, creating an instant gain of 33%. Thus, the SEC recognized that “... securities which cannot be readily sold in the public market place are less valuable than securities which can be sold ....” (SEC 1969, 3) It was not until 1971 that rigorous academic studies began to appear. The SEC (1971) conducted a detailed study of trading data on restricted stock and estimated a DLOM around 26%. Based in part on the 1971 SEC study, the Internal

1 See Abrams (2010, 301), along with Genesis 37:28 and Exodus 21:32 (Bible). See also Josephus, Antiquities, Book XII, Chapter 11, and Leviticus 27:5.
Revenue Service (1977) issued Revenue Ruling 77-287, which provides some guidance on estimating an appropriate DLOM.

Based on two different restricted stock data sources, Stockdale documented the existence of discounts from below −10% (premium) to above 90% (Stockdale 2013, v. 1, 53). Finnerty reported a range of discounts from −79% (premium) to 85% based on an analysis of 275 private placements of public equity (Finnerty 2013a, 583, Table II). Clearly, with such an enormous range of values, additional tools to address quantifying DLOM would be helpful.

In the Internal Revenue Service (IRS) publication, “Discount for Lack of Marketability Job Aid for IRS Valuation Professionals” dated September 25, 2009, page 4, the authors make the following standard definitions as well as provide a few important observations: (italics and footnotes in the original)

Marketability is defined in the International Glossary of Business Valuation Terms as “the ability to quickly convert property to cash at minimal cost.”⁵ Some texts go on to add “with a high degree of certainty of realizing the anticipated amount of proceeds.”⁶

A Discount for Lack of Marketability (DLOM) is “an amount or percentage deducted from the value of an ownership interest to reflect the relative absence of marketability.”⁷

In the alternative [non-marketable instrument], a lesser price is expected for the business interest that cannot be quickly sold and converted to cash. A primary concern driving this price reduction is that, over the uncertain time frame required to complete the sale, the final sale price becomes less certain and with it a decline in value is quite possible. Accordingly, a prudent buyer would want a discount for acquiring such an interest to protect against loss in a future sale scenario.

While there are numerous technical issues and subtle nuances, the goal of the DLOM exercise is to monetize the uncertainty surrounding the lack of marketability. Generally, it is a normative question: What ought to be the DLOM for this specific case? Obviously, there is never direct evidence for a specific case. Valuation professionals use data from numerous sources that provide indirect evidence, such as restricted stock studies. Not surprisingly, given the enormous amount of money involved in DLOM estimation, this indirect evidence often leads to vastly different valuations depending on your objective (for example, IRS or taxpayer). Also, there are many other potential discounts that are not addressed here, including minority interest discount, other transfer-ability restriction discounts, and nonsystematic risk discounts.⁸

Approaches to estimating DLOM can be generally categorized as either empirical or theoretical. Empirical approaches typically focus on market evidence from restricted stock transactions, various private placements, and private investments in public equities. One then must extrapolate from the market evidence to the particular case at hand. Given the unique attributes of every case, this extrapolation can be quite tenuous. For a concise summary of an extensive set of empirical studies, see Stockdale (2013, 47–49).

Theoretical approaches are attractive because they typically provide a parsimonious set of inputs that can be estimated for each DLOM assignment. Theoretical models are generally either based on discounted cash flow models or option valuation models. For examples of the discounted cash flow models, see Meulbroek (2005), Tabak (2002), and Stockdale (2013); see also Mercer’s quantitative marketability discount model (Stockdale 2013, 232–238).

In this paper, the model is built on existing DLOM literature related to option theory to provide a general framework for estimating DLOM that can be used in a wide array of applications. Presently, option-based DLOM approaches are rudimentary and often provide only an upper bound. An implicit problem is the underlying instrument is assumed to lack liquidity, and modern option theory relies heavily on liquid underlying instruments. This problem is not unique, as evidenced by real options applications. The framework developed here is easily modified to incorporate both risk-adjusted growth rates of the underlying instrument, as well as risk-adjusted discount rates. The focus here is not to overcome the numerous challenges of applying option theory in illiquid markets; rather, the focus is to generalize and extend existing DLOM estimation techniques already widely used in practice.

This unique approach decomposes existing models into a skill component and a hedge component. The skill component measures the DLOM attributable to the economic value lost for talented investors who suffer solely because the underlying instrument is not marketable. That is, if the underlying instrument were...
marketable, then this investor would be better off at the end of the nonmarketable period. The hedge component measures the DLOM attributable to the inability to hedge adverse market price movements solely because the underlying instrument is not marketable.

For example, the DLOM for restricted stock of a highly skilled chief executive officer (CEO) would be significantly different from the DLOM of the same restricted stock when estimating the estate taxes if this same CEO passed away. The decomposition herein provides stock when estimating the estate taxes if this same CEO significantly different from the DLOM of the same restricted skilled chief executive officer (CEO) would be significant.

The decomposition herein provides valuation professionals needed flexibility to address a wide array of DLOM valuation problems. This paper demonstrates that the model has existing option-based models as special cases.

The primary advantage of this model is the ability to explicitly address several important features common to DLOM problems. These features include the ability to externally hedge and the timing ability of the instrument owner. This model provides professionals much needed discretion and is not based on directly observable phenomenon. Unfortunately, not one single DLOM estimation approach can be based on directly observable phenomenon by its very nature. Each DLOM estimation assignment has a unique underlying instrument as well as a unique property owner. Thus, DLOM requires professional judgment, and the DLOM tools should provide this needed flexibility to reflect these judgments.

The focus here is option valuation–based monetization of the lack of marketability. The remainder of the paper is organized as follows. First, I discuss the relevant option valuation–based DLOM literature. In the next section, the option-based DLOM model is introduced, and its decomposition is explained. Then, I illustrate the model and provides alternative interpretations of some existing empirical evidence. The last section presents the conclusions.

**Option Valuation–Based DLOM Literature**

There is a vast literature that addresses estimating DLOM. For a review of this literature, as well as current DLOM estimation practices, see Stockdale (2013). This paper focuses solely on option valuation–based DLOM approaches.

**European style put option**

Chaffe (1993, 182) summarizes, “if one holds restricted or non-marketable stock and purchases an option to sell those shares at the free market price, the holder has, in effect, purchased marketability for the shares. The price of the put is the discount for lack of marketability.” The acquisition of a put option eliminates the uncertainty of the future downside risk. The shares have essentially been insured against any event that drives the price down. Recall, however, that the put option does not eliminate the benefits that accrue when the share price rises. Thus, this put option is an upper bound for DLOM, at best. Chaffe proposes using the cost of capital for the interest rate in the Black-Scholes-Merton option valuation model (BSMOV). Chaffe further illustrates the DLOM using volatilities from 60% to 90%.

The BSMOV is based on the assumption that the underlying instrument follows geometric Brownian motion, implying the terminal distribution is lognormal.7 The lognormal distribution has several well-known weaknesses, including the inability of the underlying instrument to be zero and taking on unusual features with high volatility.8 Unfortunately, DLOM is often estimated for highly volatile instruments and relatively long horizons. The terminal volatility is linear in the square root of maturity time (σ√T − t). Assuming a one-year horizon, when terminal volatility exceeds 100%, the lognormal distribution appears dubious at best. Often, estimated volatility exceeds 100%. For example, if the stock price is $100 with $100 strike price and terminal volatility of 100% (with interest rates and dividend yield assumed to be zero), then based on the lognormal distribution assumption, although the mean is $100, the median is $61, and the mode is $22 (skewness is 6.2, and excess kurtosis is 111). Given that stock returns tend to be negatively skewed, these statistics are inconsistent with observed stock price behavior. Although 100% terminal volatility seems high, it is equivalent to a four-year (6.25-year) horizon and 50% (40%) annual volatility. Therefore, DLOM estimates based on option-valuation approaches require significant professional judgment.

Several authors have extended Chaffe’s work by examining longer maturity options. See, for example, Trout (2003) and Seaman (2005a, 2005b, 2007, 2009), Barenbaum, Schubert, and Garcia (2015) combined the long put position with a short call position and analyzed DLOM based on this collar position. They examined the inherent basis risk when the options are based on a financial instrument other than the one being discounted.

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7Remember that distributions are not observable. Finance is neither physics nor mathematics—it is a human science. Future stock price changes reflect a myriad of human decisions, and any distribution is merely a conjecture of the potentials going forward.

8For more details, see Brooks and Chance (2014). One example of an unusual feature is if the stock price is $100, the expected return is 12% (annualized, continuously compounded), the standard deviation is 75% (annualized, continuously compounded), and the horizon is two years, then the mode is about $13.
Interestingly, Chaffe (1993, 182) notes “There is also a component of the discount that is related to the inability to realize an intermediate gain quickly and efficiently. For purposes of this analysis, we forego quantification of the discount factor associated with this second aspect of marketability.” Longstaff incorporates this aspect with a lookback put option.

**Lookback put option**

Longstaff (1995) captures the intermediate gain issue by assuming the investor has skill or perfect timing ability. He develops an upper bound estimate for DLOM based on a floating strike lookback put option model. Longstaff (1995, 1774) concludes, “The results of this analysis can be used to provide rough order-of-magnitude estimates of the valuation effects of different types of marketability restrictions. In fact, the empirical evidence suggests that the upper bound may actually be a close approximation to observed discounts for lack of marketability. More importantly, however, these results illustrate that option-pricing techniques can be useful in understanding liquidity in financial markets and that liquidity derivatives have potential as tools for managing and controlling the risk of illiquidity.”

I present a more general version of Longstaff’s model. Note that Longstaff assumes the lookback maximum is expressed as 

\[ M_T = \max_{0 \leq t \leq T} \{ V_t e^{r(T-t)} \} \]

where \( V_t \) denotes the underlying instrument’s value at time \( t \), the risk-free interest rate is \( r \), and the time to maturity is \( T \). Assuming the underlying instrument grows at the risk-free rate from the time of maximum value until the option maturity has the effect of removing the interest rate from the final model. I assume the more traditional case of 

\[ \hat{M}_T = \max_{0 \leq t \leq T} \{ V_t \} \]

and then consider the special case of Longstaff. Longstaff’s model is equivalent to the more traditional lookback option structure if one assumes that the risk-free interest rate is zero.

**Average-strike put option**

Finnerty (2012, 2013a) introduces an average-strike put option approach to approximating DLOM. Finnerty assumes standard dividend adjusted geometric Brownian motion, as well as assuming the instrument holder has no special skill (e.g., market-timing ability). Finnerty’s model is based on assuming an average forward price that relies on the standard carry arbitrage formula. Finnerty’s model effectively provides DLOM estimates roughly 50% lower than an equivalent plain-vanilla put option (see detailed discussion in the next section). Finnerty (2012, 67) concludes, “The average-strike put option model ... calculates marketability discounts that are generally consistent with the discounts observed in letter stock private placements, although there is a tendency to underestimate the discount when the stock’s volatility is under 45 percent or over 75 percent, especially for longer restriction periods. The marketability discounts implied by observed private placement discounts reflect[s] differences in stock price volatility, as option theory and the average-strike put option model predict.” For an interpretation of these first three models, see Abbott (2009) and Finnerty (2013b).

**Equity collar**

Barenbaum, Schubert, and Garcia (2015) document that, absent timing ability, a put option overstates DLOM. Further, if a put option can be reasonably approximated, then surely a call option can be estimated also. Thus, through a combination of a call option, a put option, and a loan, the DLOM can be approximated without the known overstatement. The key variable determining the DLOM is the interest rate spread between the appropriate lending rate and the risk-free rate. For a critique of the Barenbaum, Schubert, and Garcia (BSG) model, see Finnerty and Park (2015).

I now turn to introduce the general option-based DLOM and the flexibility to decompose it into component pieces.

**Option-Based DLOM and Decomposition**

Longstaff’s model provides both the direct measure of a plain-vanilla put option introduced by Chaffe, as well as the lookback feature measuring the economic value lost from perfect timing. Unfortunately, Longstaff’s model is merely an upper bound.

Four major issues are addressed here. First, based on Longstaff’s model, I decompose the standard floating strike lookback put option into two components, a plain-vanilla put option and a term I define as the residual lookback portion. Second, I apply this decomposition to Longstaff’s model and explore its implications by reexamining some of his results. Third, I provide a simple alternative interpretation to Finnerty’s model and introduce a weighting system. Empirically, Finnerty’s model fits some data well, but the economic intuition is difficult. Fourth, I reconfigure the BSG model, demonstrating that it provides an important alternative interpretation for the general model explored here. Finally, I present the general option-based DLOM model and explore some simple, but extreme, cases.

I have explored DLOM models where the underlying instrument lacks marketability yet option-based solutions appear reasonable. Here, I streamline the existing DLOM option-based models and provide a general framework for a wide array of applications. The key feature here is
flexibility. Unfortunately, the cost is a less parsimonious model. Professionals seeking to apply this framework can easily reduce the number of estimated parameters.

**Decomposition of floating strike lookback put option model**

Let \( \hat{L}_t(V_t, \hat{M}_t) \) denote the value at time \( t \) of a floating strike lookback put option on the underlying instrument, \( V_t \), where \( \hat{M}_t = \max_{0 \leq \tau \leq T}[V_\tau] \). One way to express the option value is based on two separate components

\[
\hat{L}_t(V_t, \hat{M}_t) = \hat{P}_t(V_t, \hat{M}_t) + \hat{L}_t(V_t, \hat{M}_t),
\]

where \( \hat{P}_t(V_t, \hat{M}_t) \) denotes the plain vanilla put portion, and \( \hat{L}_t(V_t, \hat{M}_t) \) denotes the residual lookback portion. I now examine each portion separately. The general option-based DLOM introduced later will be based on this decomposition.

**Component 1: Plain vanilla put portion (Black and Scholes 1973; Merton 1973)**

Based on the BSMOVM, we have

\[
P_t(V_t, X) = X e^{-r(T-t)} N(-d_2) - V_t e^{-\delta(T-t)} N(-d_1),
\]

where

\[
d_1 = \frac{\ln \left( \frac{V_t}{X} \right) + \left( r - \delta + \frac{\sigma^2}{2}(T-t) \right)}{\sigma \sqrt{T-t}},
\]

\[
d_2 = d_1 - \sigma \sqrt{T-t},
\]

and

\[
N(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx
\]

(standard normal cumulative distribution function).

Here,

- \( T \) time to expiration, measured in years, where the option is assumed to be evaluated at time \( t \),
- \( r \) known, annualized, continuously compounded “risk-free” interest rate,
- \( \delta \) known, annualized, continuously compounded dividend yield,
- \( \sigma \) known, annualized standard deviation of continuously compounded percentage change in the underlying instrument’s price,
- \( V_t \) observed value of the underlying instrument at time \( t \),
- \( X \) strike or exercise price, and
- \( P_t \) model value of a plain-vanilla put option.

As I will discuss later, the DLOM is influenced by dividend policy, and this model captures this influence. Note that if the put option is at the money \( (V_t = X) \) and if \( r = \delta = 0 \), then this expression can be significantly reduced to

\[
\hat{P}_t = P_t = V_t \left\{2N \left( \frac{\sigma \sqrt{T}}{2} \right) - 1 \right\}.
\]

This expression is important to understanding both the Longstaff and Finnerty models.

**Component 2: Residual lookback portion**

Based on the BSMOVM framework, the residual lookback portion can be expressed as9

\[
\hat{L}_t(V_t, \hat{M}_t) = V_t e^{-r(T-t)} \frac{\sigma^2}{2(r-\delta)}
\]

\[
\times \left[ e^{(r-\delta)(T-t)} N(d_1) - \left( \frac{V_t}{\hat{M}_t} \right)^{r-\delta/\sigma^2} N(d_3) \right]
\]

\[\quad (r \neq \delta), \quad (6a)\]

\[
\hat{L}_t(V_t, \hat{M}_t) = V_t e^{-r(T-t)} \frac{\sigma^2(T-t)}{2} N(d_1) + \sigma \sqrt{T-t} \ln(d_1)
\]

\[
+ \ln \left( \frac{V_t}{\hat{M}_t} \right) N(d_3) \]

\[\quad (r = \delta), \quad (6b)\]

where

\[
d_3 = d_1 - \frac{2(r-\delta) \sqrt{T-t}}{\sigma}, \quad \text{and} \quad (7)
\]

\[
n(d) = \frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi}}
\]

(standard normal probability density function).

Again, if the residual lookback portion is at the money \( (V_t = X) \) and \( r = \delta = 0 \), then this expression can be reduced significantly to

\[
\hat{L}_t = L_t = V_t \left[ \frac{\sigma^2(T-t)}{2} N \left( \frac{\sigma \sqrt{T-t}}{2} \right) + \sigma \sqrt{T-t} \ln \left( \frac{\sigma \sqrt{T-t}}{2} \right) \right] \quad (8)
\]

I will discuss these results in detail when I present the general option-based DLOM model. First, I highlight several new insights from Longstaff’s model in the context of this decomposition. After exploring insights related to Finnerty’s model and Longstaff’s model, I will introduce the general option-based DLOM model.

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9This well-known result can be found in Haug (2007, 142), Wilmott (2000, v. 1, 282), as well as other places. A detailed derivation of the floating strike lookback put is also available from the author.
Decomposition of Longstaff’s model

The general option-based DLOM model is based on decomposing Longstaff’s model and then applying appropriate weights to each component. Therefore, I turn now to decomposing the Longstaff model and provide important new insights along the way.

Note when \( V_t = \hat{M}_t \) (at-the-money) and \( r = \delta = 0 \) (no dividends and no underlying instrument carry cost), then \( \hat{M}_t = M_T = \max_{0 \leq s \leq T} [V_s] \), and the results are equivalent to Longstaff’s model. That is,

\[
LP_t(V_t = M_t, M_t) = P_t + L_t, \tag{9}
\]

where

\[
P_t = V_t \left[ 2N \left( \frac{\sigma \sqrt{T-t}}{2} \right) - 1 \right] \tag{10}
\]

(Longstaff’s plain-vanilla put portion), and

\[
L_t = V_t \left[ \frac{\sigma^2(T-t)}{2} N \left( \frac{\sigma \sqrt{T-t}}{2} \right) + \sigma \sqrt{T-t} \ln \left( \frac{\sigma \sqrt{T-t}}{2} \right) \right] \tag{11}
\]

(Longstaff’s residual lookback portion).

Generally, for low volatility and short horizons, roughly half of the lookback put option value is composed of the plain-vanilla put portion, and the other half is the residual lookback portion.

With longer time to maturity and higher volatilities, the residual lookback portion is significantly greater than the plain-vanilla put portion. The decomposition provided here is important because it sheds light on two important aspects of DLOM, the downside insurance provided by the plain-vanilla put option and the economic loss from the inability to incorporate investor skill. Both of these aspects can vary depending on the DLOM application. This decomposition is exploited to construct a more flexible DLOM tool. Recall, the empirical evidence seems to support Finnerty’s model as providing reasonable estimates of the economic value for the DLOM.

Alternative interpretation of Finnerty’s model

Finnerty (2012) introduced a DLOM model based on an average-strike put option. Further, he asserted “the model-predicted marketability discounts are consistent with actual private placement discounts after adjusting for the information, ownership concentration, and overvaluation effects that accompany a stock private placement” (Finnerty 2012, 53). As Finnerty’s model does not appear to fit well with other existing models, and an average-strike approach is difficult to justify, I show an alternative interpretation as well as important features of his model.

Based on the notation already presented and some rearranging, Finnerty’s DLOM model when expressed in dollars is

\[
DLOM_{Finnerty} = V_t e^{-\delta(T-t)} \left[ 2N \left( \frac{\sqrt{T-t}}{2} \right) - 1 \right], \tag{12}
\]

where

\[
v^2(T-t) = \sigma^2(T-t) + \ln \left\{ \frac{2}{\sigma^2(T-t) - \sigma^2(T-t) - 1} \right\} - 2 \ln \left( e^{\sigma^2(T-t)} - 1 \right) \left( 2 \right). \tag{13}
\]

Due to the upper limit on the volatility time term \( v^2(T-t) < \ln \left( \frac{2}{\sigma^2(T-t) - \sigma^2(T-t) - 1} \right) \), the upper limit on Finnerty’s DLOM model is (assuming \( \delta = 0 \))

\[
DLOM_{Finnerty} = 2N \left( \frac{\sqrt{\ln 2}}{2} \right) - 1 = 32.28\%.
\]

Often DLOM is observed to exceed this upper limit, so Finnerty’s model is severely limited.

Recall Finnerty assumes the residual lookback portion is zero. Consider a plain-vanilla put option with a strike price equal to the at-the-forward price \( X = F_t = V_t e^{\delta(T-t)} \), then the option can be expressed simply as

\[
P_t(V_t, X) = V_t e^{-\delta(T-t)} \left[ 2N \left( \frac{\sqrt{T-t}}{2} \right) - 1 \right].
\]

This simple plain-vanilla at-the-forward put option result is equivalent to Finnerty’s model except for the volatility term. Note that \( \sigma > v > 0 \) when volatility is positive and time to maturity is positive. Let \( \pi_v = \frac{\sigma}{\sigma} \) denote the proportion of Finnerty’s parameter when compared with the underlying instrument’s volatility. Although not shown here, except for very low volatility, short maturities and high-volatility, long maturities, the proportion term \( \left( \pi_v \right) \) is relatively stable around 57%. Based on Finnerty’s assertion regarding letter stock, then the residual lookback put portion should be zero, and only a portion of the plain-vanilla put option is reflected in the DLOM. This interpretation makes sense because a put option provides more than just marketability; it provides protection from unanticipated declines in value.

\[10\]Note that \( \ln \left\{ \frac{2}{\sigma^2(T-t) - \sigma^2(T-t) - 1} \right\} = \ln \left\{ 2 \right\} + \ln \left\{ e^{\sigma^2(T-t)} - \sigma^2(T-t) - 1 \right\} - 1 \), and the limit as \( \sigma \rightarrow 0 \) is \( \ln \left\{ e^{\sigma^2(T-t)} - \sigma^2(T-t) - 1 \right\} \). Note that \( \ln \left\{ 2 \right\} \) is a constant.

\[11\]We assume a fully arbitrated market; hence, the equilibrium forward price is as expressed above. Clearly, in markets where arbitrage activity is limited, then this forward expression is not appropriate. Most small capitalization stocks do not have an active forward market, limiting this model’s realism.
Therefore, with the proportion defined above, mathematically Finnerty’s model can simply be expressed as

$$\text{DLOM}_{\text{Finnerty}} = V_t e^{-\delta(T-t)} \left[ 2N\left( \frac{\pi_v \sigma \sqrt{T-t}}{2} \right) - 1 \right]. \quad (14)$$

Figure 1 replicates Finnerty’s Exhibit 2 with the inclusion of the plain-vanilla put option model. The results are presented in three panels for clarity. Clearly, Longstaff’s lookback put option model results in much higher discounts when compared to the other two models. Finnerty’s model is always less than the plain-vanilla put option model as his model is equivalent to using a proportionally lower volatility ($\pi_v$).

**Alternative interpretation of BSG’s model**

Barenbaum, Schubert, and Garcia proposed a DLOM estimate based on a combination of a call option, a put option, and a loan. Specifically,

$$\text{DLOM}_{\text{BSG}} = \frac{V_t - V_t e^{-k(T-t)} + C_t(V_t, X) - P_t(V_t, X)}{V_t}, \quad (15)$$

where $k$ denotes the lending rate, $C$ denotes the call option, and the strike price, $X$, is equal to the underlying value ($V_t$). Based on put-call parity,

$$\text{DLOM}_{\text{BSG}} = e^{-r(T-t)} - e^{-k(T-t)}. \quad (16)$$

Clearly, the interest rate spread ($k - r$) is the key driver for the DLOM within the BSG model.

Building on this previous theoretical work and empirical insights, I now propose a general option-based DLOM model.

**General option-based DLOM**

I introduce a weighting scheme that is general enough to handle most DLOM problems. First, the degree of external hedging opportunities will directly influence DLOM. If hedging opportunities could be pursued, then the DLOM applies only to the proportion of the underlying instrument’s volatility that cannot be hedged. Recall if 100% of the volatility can be hedged without costs, then the DLOM related solely to downside risk should be zero.\(^{12}\) That is, if a put option is available to purchase that eliminates the nonmarketable instrument’s downside risk, then a call option is likely also available to sell that will offset the cost of the put option and guarantee the future sale price, and the lending rate will equal the risk-free rate.\(^{13}\) I assume the external hedging opportunities are not specific to the investor. Clearly, if a particular investor is excluded from available hedging opportunities due to regulation or corporate culture, then this portion of the DLOM applies.

Second, the degree of investor skill will also directly influence DLOM. If a particular investor evidences some capacity for say market timing, then the lack of marketability imposes a significant expense. Clearly, perfect skill with active trading would result in near-infinite profits in a short period of time within the standard option valuation paradigm. Therefore, I propose the following general options-based DLOM (express as a percentage of the underlying instrument’s fully marketable value):

$$\text{DLOM}_{\text{BSG}} = \frac{P_t(V_t, X)}{V_t} + \frac{L_t(V_t)}{V_t}, \quad (17)$$

where $P_t$ and $L_t$ are as defined in Equations (2) and (6), respectively. Let $w_{\text{Hedge}, t}$ denote the investor-independent proportion of the underlying instrument that cannot be hedged, and let $w_{\text{Skill}, t}$ denote the investor-dependent proportion of the underlying instrument that reflects the investor’s skill (e.g., market timing).

The following are special cases of this model:

$$\text{DLOM}_{\text{Put}, t} = \frac{P_t}{V_t}, \quad (17a)$$

(European-style put),

$$\text{DLOM}_{\text{Chaffe}, t} = \frac{P_t(r = \text{Cost Of Capital})}{V_t}, \quad (17b)$$

(Chaffe),

$$\text{DLOM}_{\text{General Lookback}, t} = \frac{\hat{P}_t}{V_t} + \frac{\hat{L}_t}{V_t}, \quad (17c)$$

(general lookback),

$$\text{DLOM}_{\text{Longstaff}, t} = \frac{P_t(r = 0, \delta = 0)}{V_t} + \frac{L_t(r = 0, \delta = 0)}{V_t}, \quad (17d)$$

(Longstaff),

$$\text{DLOM}_{\text{Finnerty}, t} = \frac{P_t(\sigma = 0)}{V_t} \quad (17e)$$

(Finnerty), and

\(^{12}\)Ignoring the skill argument made below. Also, hedging always involves costs; hence, the DLOM will simply be the cost of hedging.

\(^{13}\)As previously mentioned, this is a simple illustration of the well-known put-call parity relationship. Clearly, tax consideration may result in using a zero-cost collar or other tax-efficient strategy.
Comparison of Finnerty’s Model (Fin), Plain-Vanilla Put Model (PVPut), and Longstaff’s Lookback Put Model (LLB), where 30%, 40%, and 50% Denote Input Volatility of the Underlying Instrument

**Figure 1**
Purchasing a put option is an upper bound on DLOM when the investor is not assumed to have skill. Thus, $w_{\text{Hedge},t}$ also provides an adjustment mechanism to account for this overestimate. For many, the idea that a particular investor has market timing ability is difficult. Remember, however, that $w_{\text{Skill},t}$ can always be set to zero. Also, for restricted stock of C-suite executives, it is reasonable to consider that they possess significant abilities related to financial markets. The evidence supporting skill can be found in the vast literature related to the exercise of executive stock options. See, for example, Brooks, Chance, and Cline (2012). Both weights will require professional judgment. The advantage of this model is that these two issues, hedging availability and investor skill, can be addressed independently. Previous models either ignored one feature, such as Chaffe, Finnerty, and BSG, or they embedded both features, such as Longstaff. It is a relatively simple matter to reinterpret prior empirical evidence as a means to estimate these weights. 

I now consider a few extreme examples to clarify the model. First, if the nonmarketable underlying instrument can be completely hedged, and the individual owner has no skill, then the DLOM is zero ($w_{\text{Hedge},t} = 0\%$ and $w_{\text{Skill},t} = 0\%$). Thus, even if the instrument is nonmarketable, the investor could effectively monetize their position through put, call, and/or forward transactions. Recall, if the position can be completely hedged, then the lending rate will equal the risk-free rate, and the BSG model will result in a DLOM of zero.

Second, now suppose that the nonmarketable underlying instrument cannot be hedged at all, and the individual owner has no skill, then the upper bound on DLOM is $DLOM_{t} = \frac{P}{V}$. This is an upper bound because purchasing a put option in this case compensates for more than just marketability. Note that as $k$ goes to positive infinity, then the BSG model will equal this upper bound (the call value will go to the underlying value, and the present value factor based on $k$ will go to zero).

Third, now suppose that the nonmarketable underlying instrument can be completely hedged, but the individual owner has perfect skill, though the owner can trade only once, then the upper bound on DLOM is $DLOM_{t} = \frac{C}{V}$. This DLOM estimate reflects the tremendous value afforded to those with skill. This perspective explains, in part, why some private placements occur at a premium. If

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14The variable, $w_{\text{BSG},t}$, is defined in Equation (18) below.

15More likely explanations for observed premiums include joint ventures, rapidly changing prices, and pricing date uncertainty. I thank John Stockdale for this insight.
the investor skill is unique to a particular set of underlying instruments, then the investor may be willing to pay a premium for the instrument for the opportunity to exercise their skill. The value to the investor of the position is \( V_t + L_t \), and there may be times when the particular investor cannot acquire the instrument in the traded market. I focus here solely on DLOM cases.

Fourth, now suppose that the nonmarketable underlying instrument cannot be hedged, and the individual owner has perfect skill, but can trade only once, then the upper bound on DLOM is Longstaff’s model, or with the notation used herein, \( DL_{OM_t} = \frac{D_t}{V_t} \).

Fifth, the BSG model can be expressed within this notation much like the Chaffe model. The lending rate, denoted \( k \), influences the weighting applied to the inability to hedge. Specifically,

\[
\omega_{BSG}^{\text{Hedge},t} = \frac{e^{-r(T-t)} - e^{-k(T-t)}}{P_t} V_t. \tag{18}
\]

Thus, even the BSG model can be expressed in terms of the general model presented here. Although, this weighting is a bit forced, an analysis of the DLOM following the BSG model will shed interesting light on acceptable weightings for different DLOM cases.

Finally, if DLOM is estimated using some other non-option methodology such as discounted cash flow approaches, then the option-based weight can be calculated for an independent rationality check. Though beyond the scope of the objectives here, clearly working through several option-based models will yield important insights regarding appropriate weighting schemes. For example, suppose you start with Finnerty’s model and arrive at an estimate for the DLOM. Clearly, it would be a simple exercise to estimate \( \omega_{\text{Hedge},t} \), assuming the skill component is zero. Further, based on this estimated weight, you could find the implied loan rate, \( k \), in the BSG model. Finally, you could appraise whether this loan rate is reasonable based on other observable market parameters.

I now provide a few illustrative scenarios and draw some insights from prior empirical studies.

Illustrations and Reinterpretations of Prior Empirical Evidence

Illustrative scenarios

The key advantage of the general model is the encapsulation of disparate option-based DLOM models. Hence, there is a need to familiarize ourselves with the weighting variables. Table 1 presents a detailed table of DLOM within this framework and with various weighting schemes for this model. For illustrative purposes, we assume \( r = \delta = 0\% \). Longstaff’s model (Eq. [17c] above) is the specific case of \( \omega_{\text{Hedge},t} = 100\% \), where \( \omega_{\text{Skill},t} = 100\% \) in the table (last row).

The general model presented here has the flexibility to relax the perfect timing assumption, resulting in lower and more realistic DLOMs. Note that for high volatility, longer maturities, and higher weights, the implied DLOM exceeds 100%. However, for extremely rare and mostly hypothetical cases, DLOM greater than 100% is nonsensical.\(^{16}\) The implication is that the instrument that lacks marketability becomes a liability and not an asset. The cause of this absurd result is a function of Longstaff’s model. Perfect timing ability is beyond human capacity, and, hence, based on Longstaff’s model, DLOMs will be biased high. Clearly, in almost all cases, the DLOM implying a liability is erroneous. Therefore, this weighting scheme affords a degree of flexibility needed by valuation professionals to better align with human capacity.

Experienced business valuation professionals will have a good idea of the general range for the DLOM of each valuation assignment. Table 1 provides interesting insights. For example, suppose you have a financial instrument that is not marketable for the next two years. A priori, you expect the DLOM to be around 25\%. Also, you assign no weight to the skill variable based on the particular executive’s attributes. If the appropriate parameters are as given in Table 1 with volatility equal to 60\%, then the weight applied to not hedge is 75\%. Further studies of the ability to hedge, say overall market risk via the BSG approach, will either validate your a priori view or stimulate further study, say by an application of Finnerty’s model.

Although not shown here, with positive interest rates and for longer maturities, the plain-vanilla put option actually declines in value due to the effect of discounting on put options. Also, even though the residual lookback portion is monotonically increasing with maturity, the discounting effect dampens the value of the residual lookback portion.

The plain-vanilla put option increases in value when the dividend yield increases. Dividends for DLOM are fundamentally different from dividends for exchange-traded put options. Exchange-traded options are not adjusted for cash dividends, but DLOM values applied to valuations do include the dividend over the restricted period. One easy way to handle incorporating dividends is to treat each anticipated cash dividend payment over the restricted period as a separate valuation complete with its own DLOM estimation. Thus, a quarterly pay dividend policy for a two-year restricted stock would involve

\(^{16}\)An example of a case where DLOM greater than 100% may be warranted is environmental damage caused by an unincorporated enterprise.
### Table 1

Illustration of the General Options-Based DLOM Assuming \( r = \delta = 0 \)

| Volatility (%) | 20 | 40 | 60 | 80 | 20 | 40 | 60 | 80 | 20 | 40 | 60 | 80 | 20 | 40 | 60 | 80 | 20 | 40 | 60 | 80 |
|----------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Years          | 1  | 1  | 1  | 1  | 2  | 2  | 2  | 2  | 3  | 3  | 3  | 3  | 3  | 5  | 5  | 5  | 5  | 10 | 10 | 10 | 10 |
| PV Put ($)     | 7.97 | 15.85 | 23.58 | 31.08 | 11.25 | 22.27 | 32.86 | 42.84 | 13.75 | 27.10 | 39.67 | 51.16 | 17.69 | 34.53 | 49.77 | 62.89 | 24.82 | 47.29 | 65.72 | 79.41 |
| Residual LB ($) | 9.02 | 20.28 | 34.01 | 50.44 | 13.40 | 31.46 | 54.85 | 84.17 | 17.03 | 41.28 | 73.93 | 116.04 | 23.29 | 59.19 | 110.13 | 178.15 | 36.48 | 100.23 | 197.41 | 332.40 |

*Note.* \( w(S) \) denotes \( w_{Skill\text{-}it} \); \( w(NH) \) denotes \( w_{Hedge\text{-}it} \); PV Put denotes the plain-vanilla put option model value, Residual LB denotes the residual lookback put option portion, and 1/20 denotes one year to maturity with volatility of 20.
possibly nine separate calculations (eight dividend payments and one terminal valuation) that could then be rolled up into one aggregate valuation and related DLOM.

The general option-based DLOM method presented here is flexible enough to handle the rich diversity observed in actual practice while still remaining rather parsimonious. Specifically, the ability to separate hedging and investor skill provides a more robust solution to the DLOM estimation problem. I turn now to reinterpreting some prior research.

Reinterpreting prior research

Many prior research papers have provided empirical evidence as illustrations for estimating DLOM. We review some of this work by reinterpreting their results in light of the more general model presented herein. The key insight is that the approach in this paper provides alternative interpretations for these results. With this alternative perspective, practitioners can more accurately incorporate the salient features of their particular DLOM challenge.

Dyl and Jiang (2008) examined Longstaff’s (1995) model for practical applications. They presented a specific case study where volatility is given as 60.5%, and maturity is 1.375 years. Relying on a variety of empirical sources, they opined that the DLOM should be approximately 23% and, for the sake of argument, assume they are correct. As the case study involved an estate, one would assume $\text{wSkill}_t = 0\%$.

An alternative interpretation is that $W_{\text{Hedge}}_t = 83\%$ would result in the same 23% DLOM. Thus, the general option-based approach presented here provides a reasonable way to connect option-based DLOM approaches with other existing methodologies. Applying these results within the BSG model implies a loan risk premium of approximately 4.8%.

Table 2 presents alternative perspectives on Finnerty’s (2012) Exhibit 6. Recall Finnerty’s DLOM value cannot exceed $32.28\%$ due to the volatility time parameter reaching a maximum of $\ln \left( \frac{2}{C_0} \right) = \ln(2)$, while actual DLOMs are often well in excess of $32.28\%$. Table 2 is based on the assumption that DLOM is first driven by the panel. Finnerty denotes Finnerty’s model assuming zero dividend yield. From Finnerty’s table, for each maturity ($T=2, 3,$ and 4) and Mean Model-Predicted Discount, the implied volatility was estimated. The average implied volatility of the three maturities is reported as the Implied Volatility (Finnerty). The Implied Maturity is solved such that the Mean Implied DLOM equals Finnerty’s model based on the estimated implied volatility. $W(S)$ denotes the weight applied to skill, and $W(-H)$ denotes the weight applied for the portion not hedged.

### Table 2
An Alternative Interpretation of Mean Implied DLOM

<table>
<thead>
<tr>
<th>Volatility, Range (%)</th>
<th>Mean Implied DLOM</th>
<th>Implied Volatility (Finnerty)</th>
<th>Implied Maturity (Finnerty)</th>
<th>Implied Weight, $W(S)$</th>
<th>Implied Weight, $W(-H)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Offerings Announced Prior to February 1997</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0–29.9</td>
<td>19.47%</td>
<td>24.3%</td>
<td>65.6%</td>
<td>100.0%</td>
<td></td>
</tr>
<tr>
<td>30.0–44.9</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>45.0–59.9</td>
<td>10.51</td>
<td>55.3</td>
<td>0.7</td>
<td>0.0</td>
<td>64.0</td>
</tr>
<tr>
<td>60.0–74.9</td>
<td>13.82</td>
<td>62.0</td>
<td>1.0</td>
<td>0.0</td>
<td>40.8</td>
</tr>
<tr>
<td>75.0–89.9</td>
<td>24.15</td>
<td>81.1</td>
<td>2.2</td>
<td>0.0</td>
<td>55.7</td>
</tr>
<tr>
<td>90.0–104.9</td>
<td>34.97</td>
<td>96.6</td>
<td>$+\infty$</td>
<td>0.0</td>
<td>69.2</td>
</tr>
<tr>
<td>105.0–120.0</td>
<td>61.51</td>
<td>112.6</td>
<td>$+\infty$</td>
<td>2.81</td>
<td>100.0</td>
</tr>
<tr>
<td>&gt;120.0</td>
<td>44.10</td>
<td>138.7</td>
<td>$+\infty$</td>
<td>0.0</td>
<td>65.5</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td><strong>24.50%</strong></td>
<td><strong>67.5%</strong></td>
<td><strong>3.4</strong></td>
<td><strong>0.0%</strong></td>
<td><strong>66.8%</strong></td>
</tr>
<tr>
<td>Panel B: Offerings Announced After February 1997</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0–29.9</td>
<td>12.66%</td>
<td>21.7%</td>
<td>6.8</td>
<td>40.7%</td>
<td>100.0%</td>
</tr>
<tr>
<td>30.0–44.9</td>
<td>18.56</td>
<td>33.8</td>
<td>6.7</td>
<td>31.1%</td>
<td>100.0%</td>
</tr>
<tr>
<td>45.0–59.9</td>
<td>15.92</td>
<td>52.3</td>
<td>1.9</td>
<td>0.0</td>
<td>77.2%</td>
</tr>
<tr>
<td>60.0–74.9</td>
<td>19.21</td>
<td>66.5</td>
<td>1.8</td>
<td>0.0</td>
<td>73.8%</td>
</tr>
<tr>
<td>75.0–89.9</td>
<td>21.37</td>
<td>82.7</td>
<td>1.6</td>
<td>0.0</td>
<td>66.6%</td>
</tr>
<tr>
<td>90.0–104.9</td>
<td>21.61</td>
<td>96.5</td>
<td>1.2</td>
<td>0.0</td>
<td>58.3%</td>
</tr>
<tr>
<td>105.0–120.0</td>
<td>24.89</td>
<td>111.7</td>
<td>1.3</td>
<td>0.0</td>
<td>58.8%</td>
</tr>
<tr>
<td>&gt;120.0</td>
<td>29.71</td>
<td>146.8</td>
<td>1.4</td>
<td>0.0</td>
<td>55.3%</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td><strong>21.31%</strong></td>
<td><strong>73.6%</strong></td>
<td><strong>2.0</strong></td>
<td><strong>0.0%</strong></td>
<td><strong>74.2%</strong></td>
</tr>
</tbody>
</table>

Note. This table is based on Finnerty (2012), Exhibit 6. As I do not have access to the raw data, I assumed the dividend yield was zero. Mean Implied DLOM is taken directly from Finnerty’s table. Finnerty denotes Finnerty’s model assuming zero dividend yield. From Finnerty’s table, for each maturity ($T=2, 3,$ and 4) and Mean Model-Predicted Discount, the implied volatility was estimated. The average implied volatility of the three maturities is reported as the Implied Volatility (Finnerty). The Implied Maturity is solved such that the Mean Implied DLOM equals Finnerty’s model based on the estimated implied volatility. $W(S)$ denotes the weight applied to skill, and $W(-H)$ denotes the weight applied for the portion not hedged.

We estimate the plain-vanilla put option with $r = 0\%$, $\delta = 0\%$ is equal to $4.21$ (the stock price was given at $15.1875$). The residual lookback put portion is $6.48$ for a lookback put value of $10.69$. 

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inability to hedge and only then driven by skill. Clearly, there are many other explanations possible. In only four cases, skill registered positive values, three of which occurred with low volatilities. Thus, the evidence indicates that, in general, the skill weight can safely be set to zero. Thus, Table 2 provides an illustration of the flexibility of the general option-based model presented here as well empirical evidence against investor skill.

Conclusions

A general option-based approach to estimating the DLOM is introduced in this paper. It was demonstrated to be general enough to address important DLOM challenges, including restriction period, volatility, hedging availability, and investor skill. The general option-based DLOM model was shown to contain the Chaffe model, Longstaff model, the Finnerty model, and the BSG model as special cases. The model also contains two weighting variables that provide valuation professionals much needed flexibility in addressing the unique challenges of each nonmarketable valuation assignment. Several prior results were reinterpreted based on the model presented here.

Acknowledgments

The author gratefully acknowledges the helpful comments of D. B. H. Chaffe, III, John Stockdale, Jr., John M. Griffin, Les Barenbaum, Lance Hall, Brandon N. Cline, Sang B. Kang, Kate Upton, Pavel Teterin, and Nicholas Glenn.

References


